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LETTER TO THE EDITOR

Effective medium theory for resistor networks in checkerboard geometries

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Abstract. We have considered the effective resistance of resistor networks which can be mapped onto a checkerboard geometry. Each square in the board is represented by four resistors of the same magnitude, R_i, where $R_i = R_1$ or R_2, with probabilities p_1 and $p_2 = 1 - p_1$. The configuration of the four resistors in a square can be chosen naturally in four different ways. For each of these we have calculated the effective medium theory (EMT) result for the effective resistance and compared it with the result of a numerical calculation for a large random network of the appropriate configuration. The agreement between EMT and our simulation is very good. It is worth noticing that the effective resistance falls outside the corresponding Hashin–Shtrikman bounds to the effective resistance of a continuous two-phase material. From EMT we have obtained percolation thresholds, which contain transcendental numbers (e.g., $1/\pi$).

There is much published work dealing with effective medium theories of resistor networks in which each resistor link has been assigned a resistance R_1 or R_2, with probabilities p_1 and $p_2 (p_1 + p_2 = 1)$ respectively. However, if one wants to make contact between discrete networks and continuous two-phase systems, one should let a 'grain' in the continuous case correspond to more than a single resistor in the discrete case. Figure 1 shows four natural representations when the grain is a square in a chequerboard geometry. We shall call these models *corner*, *cross*, *mid* and *side*, respectively. Consider now a checkerboard in which the squares are randomly assigned labels 1 and 2, with probabilities p_1 and p_2. In each square four resistors are placed, with the configuration a (figure 1). The resistors in a single square are all given the values R_1 or R_2, depending on the label of that square. In this way a discrete network is formed. Three other types of networks can be obtained when the resistors in each square are chosen as one of the configurations (b)–(d) (figure 1).

It is the purpose of this letter to derive the effective resistance for the networks in the effective medium theory (EMT) and to compare with numerical calculations.

The effective resistance of the random network is given by a homogeneous network with the same geometry and with a resistance R_m in every link. Standard methods (Turban 1978, Joy and Strieder 1978, Jansson and Grimvall 1985), which rely upon Thevenin's theorem for electric circuits, will be used to derive R_m.

An electric field E is applied parallel to e_1, see figure 1. This leads to a current i_0 through the link AC. We change all four resistors in a particular square to R_1 with probability p_1 and to R_2 with probability p_2. The current in the link BD is 0. The change of resistors from R_m to R_1 induces the same changes in the currents of the

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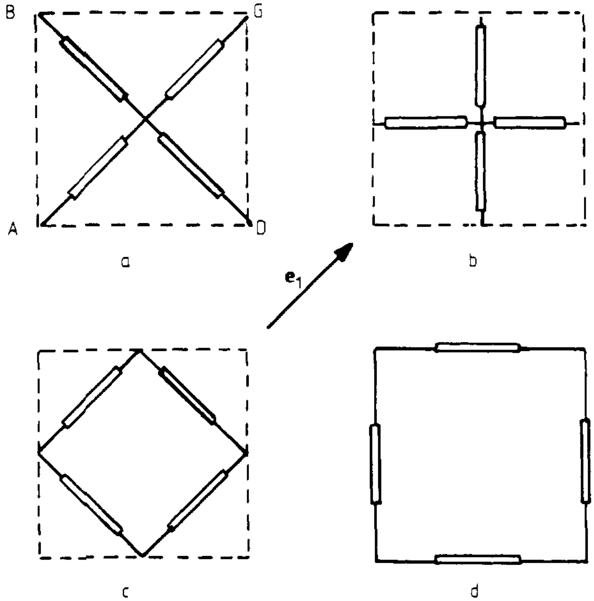

Figure 1. Discretisation models for a square: corner (a), cross (b), mid (c), side (d).

network as if an EMF equal to $\Delta u_1 = 2i_0(R_m - R_1)$ were added in the link AC, see figure 2. The new current through AC becomes $i_1 = i_0(R_z + 2R_m)/(R_z + 2R_1)$. R_z is the resistance between A and C when the four R_1 resistors in the grain are replaced by infinite resistances.

If the four resistors were changed from R_m to R_2 instead, the new current through AC would be $i_2 = i_0(R_z + 2R_m)/(R_z + 2R_2)$. In the effective medium theory, the value of R_m is chosen so that the average current is unchanged.

$$p_1 i_0 \frac{R_z + 2R_m}{R_z + 2R_1} + p_2 i_0 \frac{R_z + 2R_m}{R_z + 2R_2} = i_0. \tag{1}$$

R_z is derived by noting that the resistance between two points A and C in the homogeneous network, R_h, equals R_z in parallel with $2R_m$. R_h is found (van der Pol and Bremmer 1964) to be $2R_m(1 - 2/\pi)$. This yields $R_z = R_m(\pi - 2)$. If we take $R_z = \alpha R_m$, where $\alpha = \pi - 2$, equation (1) can be written as

$$\alpha R_m^2 + R_m[R_1(2p_2 - p_1\alpha) + R_2(2p_1 - p_2\alpha)] - 2R_1R_2 = 0. \tag{2}$$

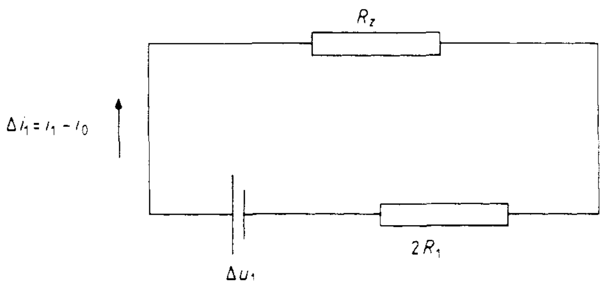

Figure 2. Equivalent network used to derive the effective medium resistance R_m.

This equation remains the same for cases 1(b)-(d); only the value of α will differ.

$$\alpha = \pi - 2 \quad \text{corner} \tag{3a}$$

$$\alpha = 2(\pi - 1) \quad \text{cross} \tag{3b}$$

$$\alpha = 4/(\pi - 2) \quad \text{mid} \tag{3c}$$

$$\alpha = 2/(\pi - 1) \quad \text{side.} \tag{3d}$$

The effective medium result of the corner model has been obtained earlier by Butcher (1975).

The four methods of discretisation can be divided into two dual pairs, corner ↔ mid and cross ↔ side. The networks generated by these models are dual to each other in the following sense (Straley 1977)

$$R(p_1, p_2)R^{D}(p_2, p_1) = R_1 R_2, \tag{4}$$

where D stands for 'duality', and $R(p_1, p_2)$ is the effective resistance of the homogeneous network. This relation also holds for our EMT values of the effective resistance. The dual network can be obtained from the original (primal) one by applying a simple duality transformation (Straley 1977).

We shall now use the EMT result, equation (2), to derive the percolation limits for the four networks described. Let $R_1 \gg R_2$. If $p_2 < \alpha/(2+\alpha)$, $R_m \approx R_1[1 - p_2(2+\alpha)/\alpha]$ and if $p_2 > \alpha/(2+\alpha)$ then $R_m \approx 2R_2/[p_2(2+\alpha) - \alpha]$. This gives a percolation threshold $p_{2c} = \alpha/(2+\alpha)$. The duality relation (3) gives the percolation threshold for the dual network $p_{2c}^{D} = 1 - p_{2c}$. The percolation thresholds are

$$p_{2c} = 1 - 2/\pi \quad \text{corner} \tag{5a}$$

$$p_{2c} = 1 - 1/\pi \quad \text{cross} \tag{5b}$$

$$p_{2c} = 2/\pi \quad \text{mid} \tag{5c}$$

$$p_{2c} = 1/\pi \quad \text{side.} \tag{5d}$$

In our numerical calculations we generated large random networks (200×200 resistors) of types (a)-(d) (see figure 1). The effective resistance was obtained by an iterative method which minimises the entropy production (Jansson and Grimvall 1985). The EMT results and those given by our numerical calculations are shown together with the Hashin-Shtrikman (HS) bounds (Hashin and Shtrikman 1962) in figure 3. The HS bounds apply to the effective resistivity of statistically isotropic and homogeneous continuous materials which are two-dimensional and consist of two phases. We have identified p_1 with the surface fraction of phase 1, and R_1 with the resistivity ρ_1 of phase 1. Perhaps the most striking result is that all four models, as well as numerical results, violate the Hashin-Shtrikman bounds. It indicates that a discretisation using few resistors to represent a grain may be quite misleading. This result will be dealt with in detail in a forthcoming paper.

We have established the effective resistance and the percolation concentrations of four resistor networks. These have been constructed using methods which emerge naturally from a discretisation of the disordered checkerboard geometry.

The effective medium theory gives an effective resistance which is in good agreement with our results from an iterative numerical calculation on random networks of the appropriate geometry. The approximate percolation limits for the concentration p_2, as derived from the EMT result, equation (2), are 0.36, 0.68, 0.64, 0.32, for the four

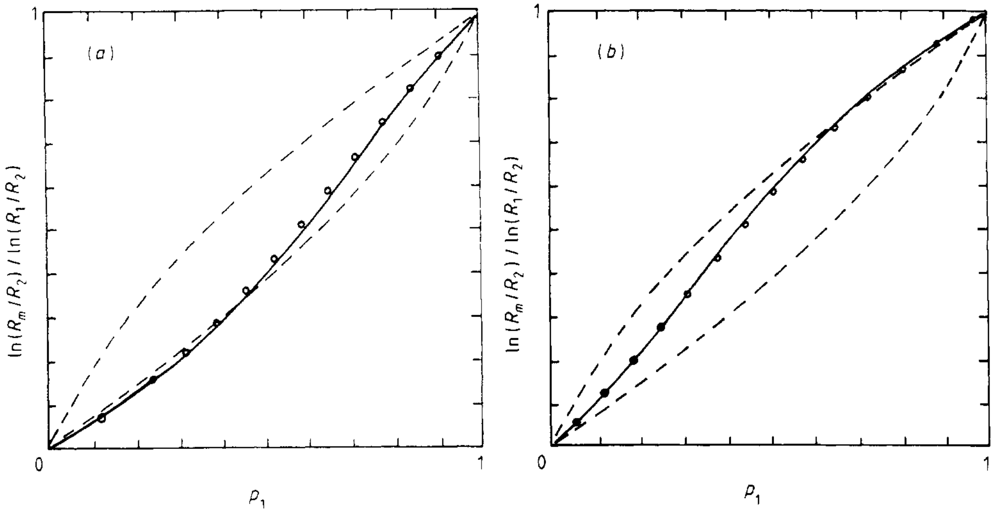

Figure 3. Effective resistance, R_m, as given by EMT (—) and numerical calculations (○), for the random networks generated by the corner (a) and cross (b) models, applied to an irregular checkerboard. The Hashin-Shtrikman bounds (- - -) are given for comparison. $R_1/R_2 = 10$.

networks, respectively. This should be compared with $p_c = 0.5$ for the continuous case (Dykhne 1970).

It should be noticed that the results of the EMT for our discrete resistor networks contain the quantity π. Neither the EMT results for the discrete bond model (Jansson and Grimvall 1985) nor EMT results for continuous materials (Landauer 1978, Bruggeman 1935) contain transcendental numbers. This indicates the non-trivial nature of our netork problem.

Both the EMT and the numerical calculations yield results for the effective resistance, which violate the corresponding Hashin-Shtrikman bounds to the resistance in the continuous case.

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